Online Exam Statistical Reasoning 2020/2021 - Part 1/3

Date: Thursday, April 1, 2021
Time: 15:00-18:00
Place: Nestor Online Exam
Progress code: WBMA038-05

Rules to follow:

- This is an open book online exam.
- You are allowed to consult: Textbooks, written and printed notes, all files on your computer disc and all files on Nestor.
- You are **neither** allowed to communicate with anyone about the exercises **nor** to use the internet to search for possible solutions.
- To prevent fraud, this online exam comes in three parts. And you have one hour to work on each part.
- This is the 1st exam part. It contains the Exercises 1-2.
- The 2nd and 3rd part will be made available at 16h and 17h, respectively.
- In each part you can reach up to 30 points; i.e. 90 points in total.
- The deadline for uploading your solutions for this 1st part is 16:05h.
- **We wish you success with the completion of the three exam parts!**

START OF EXAM - PART 1/3

1. **Posterior distribution and marginal likelihood.** **20**
Consider a random (iid) sample of discrete random variables $Y_1, \dots, Y_n$ with density:

$$p(y|\theta) = \theta \cdot (1-\theta)^y \qquad (y \in \mathbb{N}_0)$$

where $\theta \in [0,1]$ has a BETA(a,b) prior distribution with density:

$$p(\theta|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \qquad (\theta \in [0,1])$$

where $a > 0$ and $b > 0$ are known hyperparameters.

(a) 8+2 Derive the posterior distribution of θ and interpret the hyperparameters a and b in terms of 'pseudo counts'.

(b) 10 Derive the marginal likelihood $p(y_1, \dots, y_n)$

2. **Power posterior distribution.** **10**
Consider a random (iid) sample $Y_1, \dots, Y_n$ from a Poisson distribution with density

$$p(y|\theta) = \frac{\theta^y \cdot e^{-\theta}}{y!} \qquad (y \in \mathbb{N}_0)$$

where $\theta > 0$ is an unknown parameter, on which we impose a prior distribution with density:

$$p(\theta) = \lambda \cdot e^{-\lambda\theta} \qquad (\theta \in \mathbb{R}^+)$$

where $\lambda > 0$ is a known hyperparameter.

(a) 2 What is the relationship between the Exponential distribution and the Gamma distribution?

(b) 8 Derive the power posterior distribution of θ, for the given inverse temperature $\tau \in [0,1]$.

RECALL:
The density of a Gamma distribution with parameters $a > 0$ and $b > 0$ is:

$$p(x|a,b) = \frac{b^a}{\Gamma(a)} \cdot x^{a-1} \cdot \exp\{-b \cdot x\} \qquad (x \in \mathbb{R}^+)$$

Online Exam Statistical Reasoning 2020/2021 - Part 2/3

Rules to follow:

- This is the 2nd exam part. It contains the Exercises 3-4.
- The 3rd part will be made available at 17h.
- The deadline for uploading your solutions for this 2nd part is 17:05h.
- **We wish you success with the completion of the three exam parts!**

START OF EXAM - PART 2/3

3. **Gaussian and Gamma distributions.** $\boxed{\mathbf{20}}$
Let X be a random variable with sample space $\mathbb{R}$ and density $p(x)$.
What is the distribution of X when...

   (a) $\boxed{5}$ $p(x) \propto \exp\{-2x^2 - 4x\}$

   (b) $\boxed{5}$ $p(x) \propto 2 \cdot \exp\{-\frac{1}{2}x - 2\}$

   Now assume that the density of a random variable λ^{-1} with sample space $\mathbb{R}^+$ is proportional to the density of the n-dimensional Gaussian with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma} = \lambda\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

   (c) $\boxed{5}$ What is then the distribution of λ?

   Now consider the 2-dimensional random vector $\mathbf{X} = (X_1, X_2)^T$. What is the distribution of $\mathbf{X}$ when we have for its bivariate density:

   (d) $\boxed{5}$ $p(\mathbf{x}) = p(x_1, x_2) \propto \exp\{-2x_1^2 - 4x_2^2 - 4x_1x_2\}$

   <u>RECALL:</u>
   The univariate Gaussian distribution with mean μ and variance σ^2 has the density:

$$p(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\{-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}\} \qquad (x \in \mathbb{R})$$

   The n-dimensional Gaussian with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ has the density:

$$p(\mathbf{x}|\boldsymbol{\mu}, \mathbf{\Sigma}) = (2\pi)^{-n/2} \cdot \det(\mathbf{\Sigma})^{-1/2} \cdot \exp\{-\frac{1}{2} \cdot (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\} \qquad (\mathbf{x} \in \mathbb{R}^n)$$

   The density of a Gamma distribution with parameters $a > 0$ and $b > 0$ is:

$$p(x|a, b) = \frac{b^a}{\Gamma(a)} \cdot x^{a-1} \cdot e^{-b \cdot x} \qquad (x > 0)$$

   For an n-by-n matrix A and $\alpha \in \mathbb{R}$ it holds:

$$\det(\alpha \cdot A) = \alpha^n \cdot \det(A)$$

4. **Gaussian marginal likelihood.** **10**

Consider two Gaussian distributed random variables Y_1 and Y_2 with:

$$Y_i|\mu \sim \mathcal{N}(0.5\mu, 0.5) \qquad (i = 1, 2)$$

with the prior distribution: $\mu \sim \mathcal{N}(2, 2)$.

Compute:

(a) 2 the marginal distribution of Y_i $(i = 1, 2)$

(b) 3 the bivariate conditional distribution of $(Y_1, Y_2)|(\mu = 2)$

(c) 5 the bivariate marginal distribution of (Y_1, Y_2)

RECALL: The following marginalization rule:
Let $\mathbf{y}$ be an N-dimensional random vector, $\boldsymbol{\beta}$ be a k-dimensional random vector, $\mathbf{X}$ be a fixed N-by-k matrix, and $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ be fixed positive definite matrices. When

$$\begin{aligned} \mathbf{y}|\boldsymbol{\beta} &\sim \mathcal{N}_N(\mathbf{X}\boldsymbol{\beta},\ \boldsymbol{\Sigma}_1) \\ \boldsymbol{\beta} &\sim \mathcal{N}_k(\boldsymbol{\mu},\ \boldsymbol{\Sigma}_2) \end{aligned}$$

then marginalization over $\boldsymbol{\beta}$ yields:

$$\mathbf{y} \sim \mathcal{N}_N(\mathbf{X}\boldsymbol{\mu},\ \boldsymbol{\Sigma}_1 + \mathbf{X}\boldsymbol{\Sigma}_2\mathbf{X}^T)$$

Online Exam Statistical Reasoning 2020/2021 - Part 3/3

Rules to follow:

- This is the 3rd exam part. It contains the Exercises 5-6.
- The deadline for uploading your solutions for this 3rd part is 18:05h.
- **We wish you success with the completion of the three exam parts!**

START OF EXAM - PART 3/3

5. **Linear Regression.** **20**

Consider a linear regression model with response vector $\mathbf{y} \in \mathbb{R}^n$, design matrix $\mathbf{X} \in \mathbb{R}^{n,k+1}$, and regression parameter vector $\boldsymbol{\beta} \in \mathbb{R}^{k+1}$:

$$\mathbf{y}|\boldsymbol{\beta} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\Sigma}$ is a known positive definite n-by-n matrix.
On $\boldsymbol{\beta}$ we impose a prior distribution whose density fulfills:

$$p(\boldsymbol{\beta}) \quad \propto \quad \exp\{-\boldsymbol{\beta}^T\boldsymbol{\beta}\}$$

(a) 5 Conclude what the prior distribution of $\boldsymbol{\beta}$ is.

(b) 15 Compute the posterior distribution of $\boldsymbol{\beta}$.

RECALL:
The multivariate Gaussian distribution $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ has density:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-n/2} \cdot \det(\boldsymbol{\Sigma})^{-1/2} \cdot \exp\{-\frac{1}{2} \cdot (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\} \qquad (\mathbf{x} \in \mathbb{R}^n)$$

6. **Multivariate Gaussian distribution.** **10**

Consider a random (iid) sample $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ from an N-dimensional Gaussian distribution:

$$\mathbf{Y}_i|\boldsymbol{\mu} \sim \mathcal{N}_N(\boldsymbol{\mu}, \mathbf{I}) \qquad (i = 1, \ldots, n)$$

where $\mathbf{I}$ is the N-by-N identity matrix and $\boldsymbol{\mu} \in \mathbb{R}^N$ is an unknown mean vector, on which we impose an 'improper' prior, whose density fulfills:

$$p(\boldsymbol{\mu}) \propto c$$

where $c > 0$ is a constant.

Exercise: Derive the posterior distribution of $\boldsymbol{\mu}$.

RECALL:
The multivariate Gaussian distribution $\mathcal{N}_N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ has density:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-N/2} \cdot \det(\boldsymbol{\Sigma})^{-1/2} \cdot \exp\{-\frac{1}{2} \cdot (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\} \qquad (\mathbf{x} \in \mathbb{R}^N)$$